

Time functions as utilities

E. Minguzzi

Dipartimento di Matematica Applicata, Università degli Studi di Firenze, Via S. Marta 3,
I-50139 Firenze, Italy
E-mail: ettore.minguzzi@unifi.it

Abstract: Every time function on spacetime gives a (continuous) total pre-ordering of the spacetime events which respects the notion of causal precedence. The problem of the existence of a (semi-)time function on spacetime and the problem of recovering the causal structure starting from the set of time functions are studied. It is pointed out that these problems have an analog in the field of microeconomics known as utility theory. In a chronological spacetime the semi-time functions correspond to the utilities for the chronological relation, while in a K -causal (stably causal) spacetime the time functions correspond to the utilities for the K^+ relation (Seifert's relation). By exploiting this analogy, we are able to import some mathematical results, most notably Peleg's and Levin's theorems, to the spacetime framework. As a consequence, we prove that a K -causal (i.e. stably causal) spacetime admits a time function and that the time or temporal functions can be used to recover the K^+ (or Seifert) relation which indeed turns out to be the intersection of the time or temporal orderings. This result tells us in which circumstances it is possible to recover the chronological or causal relation starting from the set of time or temporal functions allowed by the spacetime. Moreover, it is proved that a chronological spacetime in which the closure of the causal relation is transitive (for instance a reflective spacetime) admits a semi-time function. Along the way a new proof avoiding smoothing techniques is given that the existence of a time function implies stable causality, and a new short proof of the equivalence between K -causality and stable causality is given which takes advantage of Levin's theorem and smoothing techniques.

1. Introduction

On the spacetime (M, g) we write as usual $p < q$ if there is a future directed causal curve connecting p to q , and write $p \leq q$ if $p < q$ or $p = q$. The causal relation is given by $J^+ = \{(p, q) \in M \times M : p \leq q\}$. For fixed time orientation

these notions depend only on the the class \mathbf{g} of metrics conformal to g . Once the causal relation is defined it is possible to define a *time function* as a continuous function $t : M \rightarrow \mathbb{R}$ such that if $p < q$ then $t(p) < t(q)$. In other words a time function is defined all over the spacetime, it is continuous, and increases over every causal curve. A time function which is C^1 with a past directed timelike gradient is a *temporal function*. Following [42], a semi-time function is a continuous function $t : M \rightarrow \mathbb{R}$ such that if $p \ll q$ then $t(p) < t(q)$ (note that by continuity we have also $(p, q) \in \overline{I^+} \Rightarrow t(p) \leq t(q)$).

These definitions clarify that in the framework of general relativity the notion of causality is more fundamental than that of time. Indeed, not all the spacetimes admit a time function. A spacetime admits a time function iff it admits a temporal function iff it is stably causal [18, 19, 4]. The history of this result is quite interesting.

In order to prove the existence of a time function Geroch [17] suggested to introduce a positive measure μ on spacetime so that M has unit measure (in fact this measure has to be chosen so as to satisfy some admissibility constraints [12]), and to define $t^-(p) = \mu(I^-(p))$. In globally hyperbolic spacetimes, and actually in causally continuous ones [20, 12, 35], the idea works in fact t^- can be shown to be continuous. Nevertheless, these causality conditions are stronger than stable causality and without them t^- is only lower semi-continuous.

The proof that stable causality implies the existence of a time function was obtained by Hawking through a nice averaging technique [18, 19]. In short he noted that if the spacetime is stably causal then there is a one parameter family of metrics with cones strictly larger than g , $g_\lambda > g$ with $\lambda \in [1, 2]$, $\lambda < \lambda' \Rightarrow g_\lambda < g_{\lambda'}$, so that (M, g_λ) is causal. He then defined $t(p) = \int_1^2 t_\lambda^-(p) d\lambda$ where the function t_λ^- is defined as before but with respect to the metric g_λ . He was then able to prove the continuity of t (see [19]).

The proof of the converse presents several difficulties particularly because a time function t for (M, g) need not be a time function for some (M, g') with $g' > g$, consider for instance $t = x^0 - \tanh x^1$ in the 1+1 Minkowski spacetime of metric $g = -(dx^0)^2 + (dx^1)^2$. Nevertheless, the proof that a temporal function implies stable causality is easy [19] and thus there remained the issue of proving that the existence of a time function implies the existence of a temporal function. This smoothability problem was considered by Seifert [42] but his arguments were unclear. A rigorous proof was finally given by Bernal and Sánchez in [4].

Further insight to the problem of the existence of time come from the relational approach to causality. Stable causality can be shown to be equivalent to the antisymmetry of the Seifert relation [41] $J_S^+ = \bigcap_{g' > g} J_{g'}^+$ (a rigorous proof can be found in [20] and [31]). The nice feature of this relation is that it is both closed and transitive whereas J^+ has only the latter property and $\overline{J^+}$ has only the former. In fact one may ask if J_S^+ is the smallest relation containing J^+ with this property. The answer is negative unless some causality conditions are added [31]. Therefore, it is natural to introduce the relation K^+ defined as the smallest closed and transitive relation which contains J^+ (see [44]). The spacetime is said to be K -causal if K^+ is antisymmetric. Recently [34], I have proved the equivalence between K -causality and stable causality and that if K -causality holds then $K^+ = J_S^+$.

The previous result shows that the antisymmetry of K^+ implies stable causality and hence the existence of a time function. It seems reasonable to expect that

(i): this theorem depends only on the transitivity and closure properties of K^+ , and that therefore passing through stable causality should not be essential. (ii): The existence of a time function should imply K -causality (or stable causality) directly without using the smoothing argument. Finally, given the fact that K -causality implies the existence of a time function one would like to prove that (iii): under stable causality the set of time functions allowed by the spacetime can be used to recover the relation K^+ .

In a first version of this work I presented proofs for points (ii) and (iii) but then searching for fundamental results using only the closure of a relation in connection with problem (i), I discovered a large body of literature in utility theory with important implications for causality (most articles were published in economics journals). In fact the problem of the existence of a utility function in a set of alternatives for an individual is formally similar to that of the existence of time. Surprisingly, these results have been totally overlooked by relativists. In the next section I summarize this long parallel line of research which will be used to draw implications for causality theory and in particular for the problem of the existence of time.

I refer the reader to [35,30] for most of the conventions used in this work. In particular, I denote with (M, g) a C^r spacetime (connected, time-oriented Lorentzian manifold), $r \in \{3, \dots, \infty\}$ of arbitrary dimension $n \geq 2$ and signature $(-, +, \dots, +)$. On $M \times M$ the usual product topology is defined. All the causal curves that we shall consider are future directed. The subset symbol \subset is reflexive, $X \subset X$.

2. Preorders and utility theory

Recall¹ that a binary relation $R \subset X \times X$ on a set X is called a *preorder* if it is reflexive and transitive, a *strict partial order* if it is irreflexive and transitive, and an *equivalence relation* if it is reflexive, transitive and symmetric. A preorder which satisfies the antisymmetry property $(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$, is a *partial order*. The preorder or strict partial order R such that $x \neq y \Rightarrow (x, y) \in R$ or $(y, x) \in R$, is *complete*. The property, if $a, b \in X$ then $(a, b) \in R$ or $(b, a) \in R$ is the *totality* property and is equivalent to completeness and reflexivity. A preorder which satisfies the totality property, is a *total preorder*. A preorder which respects both the totality and the antisymmetry property is a *total order*. A strict partial order which is complete is a *complete order*.

For short we also write² $x \leq_R y$ if $(x, y) \in R$; $x \sim_R y$ if $(x, y) \in R$ and $(y, x) \in R$; and $x <_R y$ if $x \leq_R y$ and not $x \sim_R y$. The relation \sim_R is an equivalence relation called the *equivalence relation part* while $<_R$ is a strict partial order called the *strict partial order part*. Their union gives R , i.e. $x \leq_R y$ iff $x \sim_R y$ or $x <_R y$. Note that if R is a partial order then $<_R$ is obtained from R by removing the diagonal $\Delta = \{(x, x) : x \in X \times X\}$, and conversely R is obtained by adding the diagonal to $<_R$, that is, R is the smallest reflexive relation containing $<_R$ (the reflexive closure).

¹ Unfortunately, in the literature there is no homogeneous terminology, so that used in this paper differs from that of many cited articles.

² Unfortunately, if $R = J^+$ then while \leq_{J^+} has the same meaning of the symbol \leq in relativity, the relation $<_{J^+}$ coincides with $<$ only in causal spacetimes.

Given a (total) preorder R , the quotient X/\sim_R endowed with the induced order $[p] \leq_R [q]$ iff $p \leq_R q$, is a partial (resp. total) order.

As usual we denote $R^+(x) = \{y \in X : (x, y) \in R\}$ and $R^-(x) = \{y \in X : (y, x) \in R\}$.

We say that R_2 extends R_1 if $x \sim_{R_1} y \Rightarrow x \sim_{R_2} y$ and $x <_{R_1} y \Rightarrow x <_{R_2} y$.

Through a rather simple application of Zorn's lemma Szpilrajn [45] proved

Theorem 1. (*Szpilrajn*) *Every strict partial order can be extended to a complete order. Moreover, every strict partial order is the intersection of all the complete orders that extend it. (by adding the diagonal one has a corresponding statement for partial orders extended by total orders.)*

The former statement in the theorem is also known as the *order extension principle*. The latter statement is sometimes attributed to Dushnik and Miller [14], however, although not stated explicitly in [45], it is a trivial consequence of a remark in Szpilrajn's paper.

Since to any preorder one can associate a partial order passing to the quotient with respect to the equivalence relation \sim_R , it is easy to prove from theorem 1 the following [13,6]

Theorem 2. *Every preorder can be extended to a total preorder. Moreover, every preorder is the intersection of all the total preorders that extend it.*

I note that the proof in [6] is such that the total extension can be chosen (or restricted in the second part) in such a way that the 'indifference sets' $[p]$ are not enlarged passing from the preorder R to its total extension C , i.e. $x \sim_R y \Leftrightarrow x \sim_C y$.

These results were taken as reference for many other developments, and in fact have been generalized in several directions [1].

Meanwhile, in microeconomics the preference of an individual for a set of alternatives or prospects X was modeled as a *total preorder* R on X . The idea was that an individual is able to tell whether one option or the other is preferred. These preferences were quantified by an *utility function*. An utility for a transitive relation R is a function $u : X \rightarrow \mathbb{R}$ with the *strictly isotone*³ property (citare birkoff) namely that⁴

$$"x \sim_R y \Rightarrow u(x) = u(y)" \text{ and } "x <_R y \Rightarrow u(x) < u(y)." \quad (1)$$

Often on X one has a topology which makes rigorous the idea that an alternative is similar or close to another. In this case one would like to have a *continuous* utility function otherwise the closeness of the alternatives would not be correctly represented by the utility. Eilenberg [15] and Debreu [10,11] (see also [39,27]) were able to prove, under weak topological assumptions, that a continuous utility exists provided $R^-(x)$ and $R^+(x)$ are closed for every $x \in X$.

With the work of Aumann [2] and other economists it became clear that the assumption of totality was too restrictive. It turns out that it is unreasonable to assume that the individual is able to decide the preference for one of two alternatives. This conclusion is even more compelling if one models a group of

³ A function is isotone if $(x, y) \in R \Rightarrow u(x) \leq u(y)$. Note that constant functions are isotone.

⁴ For a total preorder this definition can be replaced by: $x \leq_R y$ if and only if $u(x) \leq u(y)$.

persons rather than an individual. In order to include some indecisiveness the space of alternatives X has to be endowed with a preorder, the totality condition being removed. The previous results on the existence of a continuous utility must therefore be generalized, and there is indeed a large literature on the subject. The reader is referred to the monograph by Bridges and Mehta for a nice reasoned account [7].

The problem is of course that of finding natural conditions which imply the continuity of the utility function. We already suspect, given the suggestions from the spacetime problem above, that this property is the closure of the relation R in $X \times X$, $R = \bar{R}$ (sometimes called *continuity* in the economics literature). In fact, it can be easily shown [46] that for a total preorder this property coincides with that used in Eilenberg and Debreu theorem, namely: $R^-(x)$ and $R^+(x)$ are closed for every $x \in X$ (sometimes called *semicontinuity* in the economics literature).

In the literature many other direction have been explored but as we shall see the closure condition has given the most powerful results. Considerably important has proved the work by Nachbin [36], who studied in deep closed preorders on topological spaces and obtained an extension to this domain of the Urysohn separation and extension theorems. These results were used to obtain new proofs of the Debreu theorem [27], and over them have been built the subsequent generalizations. For our purposes, the final goal was reached by Levin [25, 7] who proved

Theorem 3. (Levin) *Let X be a second countable locally compact Hausdorff space, and R a closed preorder on X , then there exists a continuous utility function. Moreover, denoting with \mathcal{U} the set of continuous utilities we have that the preorder R can be recovered from the continuous utility functions, namely there is a multi-utility representation*

$$(x, y) \in R \Leftrightarrow \forall u \in \mathcal{U}, u(x) \leq u(y). \quad (2)$$

Curiously, as it happened for Szpilrajn's theorem, the second part of this statement was not explicitly given in Levin's paper. Nevertheless, it is a trivial consequence of his proof that if $x \not\leq_R y$ then there is a continuous utility function such that $u(x) > u(y)$ (see the end of the proof of [7, Lemma 8.3.4]). Apparently, this representation possibility has been pointed out only quite recently in a preprint by Evren and Ok [16].

Clearly, Levin's theorem can be regarded as the continuous analog of the Szpilrajn's theorem.

It is curious to note that while there was, as far as I know, no communication between the communities of relativists and economists these two parallel lines of research passed through the very same concepts. For instance, Sondermann [43] (see also [8]) introduced a measure on X and built an increasing function exactly as Geroch did, even more his admissibility requirements for the measure were close to those later introduced by Dieckmann [12] in the relativity literature. As expected he could only obtain lower semi-continuity for the utility, indeed we know that Geroch's time is continuous only if some form of reflectivity is imposed [20].

Among the theorems which do not use in an essential way some closure assumption, which in most cases can be deduced from Levin's theorem, a special mention deserves Peleg's theorem which instead uses an openness condition [38].

According to Peleg a *strict* partial order S is *separable* if there is a countable subset C of X such that for any $(x, y) \in S$ the diamond $S^+(x) \cap S^-(y)$ contains some element of the subset C , and *spacious* if for $(x, y) \in S$, $\overline{S^-(x)} \subset S^-(y)$.

Theorem 4. (Peleg) *Let S be a strict partial order on a topological space X . Suppose that (a) $S^-(x)$ is open for every $x \in X$, (b) S is separable, and (c) S is spacious, then there is a function $u : X \rightarrow \mathbb{R}$ such that $(x, y) \in S \Rightarrow u(x) < u(y)$.*

Note that u is an utility in the sense of Eq. (1). It has been clarified that Debreu's theorem can be regarded as a consequence of Peleg's [24, 28], while the relation with Levin's theorem is less clear (but see [21, 22]). We will be able to say more on that in the next section where we shall apply the previous theorems to the spacetime case.

3. Application of utility theory to causality

Our strategy will be that of applying Peleg's theorem to the open relation I^+ and Levin's theorem to the closed relation K^+ . As we shall see the semi-time functions and the time functions correspond to the utilities for the relations I^+ and K^+ respectively, provided they are antisymmetric.

We start with the former case

3.1. Semi-time functions and I^+ -utilities. In this section we apply Peleg theorem by setting $X = M$, the spacetime manifold, and $S = I^+$. The assumption that I^+ is a strict partial order is equivalent to chronology and conditions (a) and (b) of Peleg's theorem are satisfied. Note that in a chronological spacetime a continuous utility for I^+ is a continuous function t such that $x \ll y \Rightarrow t(x) < t(y)$, thus the continuous utilities for I^+ are exactly the semi-time functions. Therefore we have only to understand the condition of spaciousness: $x \ll y \Rightarrow \overline{I^-(x)} \subset I^-(y)$. This problem is answered by the following

Lemma 1. *Let (M, g) be a spacetime, then the spaciousness condition, $x \ll y \Rightarrow \overline{I^-(x)} \subset I^-(y)$, is equivalent to the future reflectivity condition, $p \in \overline{I^-(q)} \Rightarrow q \in I^+(p)$.*

(The previous definition of future reflectivity is equivalent to the usual one $I^-(w) \subset I^-(z) \Rightarrow I^+(z) \subset I^+(w)$, see [20, 35].)

Proof. Assume that (M, g) is future reflective, take $x \ll y$ and let $z \in \overline{I^-(x)}$ then $x \in \overline{I^+(z)}$ and since $I^-(y)$ is open, $z \ll y$. As z is arbitrary $\overline{I^-(x)} \subset I^-(y)$.

Conversely, assume the spacetime is spacious. Let $p \in \overline{I^-(q)}$ and take $r \in I^+(q)$ then by spaciousness, $I^-(r) \supset \overline{I^-(q)} \ni p$ so that $r \in I^+(p)$. As r can be chosen arbitrarily close to q , we have $q \in \overline{I^+(p)}$. \square

With these preliminaries, Peleg's theorem becomes (in the statement we include the past version)

Theorem 5. *A chronological, future or past reflective spacetime admits a semi-time function.*

This theorem is new in causality theory as there were no previous results establishing the existence of a semi-time function. We observe that if the causality assumption were somewhat stronger, with chronology replaced by distinction ($I^+(x) = I^+(y)$ or $I^-(x) = I^-(y) \Rightarrow x = y$) then the spacetime would be K -causal [30, Theorem 3.7], thus it would admit a time function.

Also note that if future or past reflectivity is strengthened to reflectivity then the chronological assumption can be weakened to non-total viciousness (namely the chronology violating set does not coincide with M). This fact follows from a theorem by Clarke and Joshi which states that a non-totally vicious spacetime which is reflective is chronological [9, Prop. 2.5] (see also [23]).

In [34] I have introduced the *transverse conformal ladder* and proved that future or past reflectivity implies the transitivity of $\overline{J^+}$, that is $K^+ = \overline{J^+}$, (see the proof of theorem 3 in [34]). Thus one could try to strengthen the previous theorem by replacing ‘future or past reflectivity’ with the transitivity of $\overline{J^+}$. As we shall see in the next section it is indeed possible to do that.

3.2. Time functions and K^+ -utilities. Any spacetime is a paracompact Hausdorff manifold and as such it satisfies the topological conditions of Levin’s theorem (in fact it even admits a complete Riemannian metric [37]). Since we wish to apply this theorem to the relation K^+ we have first to establish the relation between the time functions and the continuous K^+ -utilities.

In this section we shall prove, without using smoothing techniques [4] or the equivalence between K -causality and stable causality [34], that a spacetime is K -causal if and only if it admits a time function. Along the way we shall also prove that in a K -causal spacetime the K^+ -utilities are exactly the time functions.

Lemma 2. *A spacetime which admits a time function t is strongly causal.*

Proof. The proof of [30, Theorem 3.4] shows that if (M, g) is not strongly causal then there are points $x, c \in M$ such that $x < c$ and $(c, x) \in \overline{J^+}$. Since for any pair $(p, q) \in J^+$ it is $t(q) - t(p) \geq 0$ and t is continuous, $t(x) < t(c) \leq t(x)$, a contradiction. \square

Recall that a spacetime is non-total imprisoning if no future inextendible causal curve is contained in a compact set (replacing *future* with *past* gives the same property [3, 33]). Strong causality implies non-total imprisonment [19].

Lemma 3. *Let (M, g) be non-total imprisoning. Let $(p, q) \in K^+$ then either $(p, q) \in J^+$ or for every relatively compact open set $B \ni p$ there is $r \in \dot{B}$ such that $p < r$ and $(r, q) \in K^+$.*

Proof. Consider the relation

$$R^+ = \{(p, q) \in K^+ : (p, q) \in J^+ \text{ or for every relatively compact open set } B \ni p \text{ there is } r \in \dot{B} \text{ such that } p < r \text{ and } (r, q) \in K^+\}.$$

It is easy to check that $J^+ \subset R^+ \subset K^+$. We are going to prove that R^+ is closed and transitive. From that and from the minimality of K^+ it follows $R^+ = K^+$ and hence the thesis.

Transitivity: assume $(p, q) \in R^+$ and $(q, s) \in R^+$. If $(p, s) \in J^+$ there is nothing to prove. Otherwise we have $(p, q) \notin J^+$ or $(q, s) \notin J^+$. Let $B \ni p$ be an open relatively compact set.

If $(p, q) \notin J^+$ there is $r \in \dot{B}$ such that $p < r$ and $(r, q) \in K^+$, thus $(r, s) \in K^+$ and hence $(p, s) \in R^+$.

It remains to consider the case $(p, q) \in J^+$ and $(q, s) \notin J^+$. If $p = q$ then $(p, s) = (q, s) \in R^+$. Otherwise, $p < q$ and if $q \notin B$ the causal curve γ joining p to q intersects \dot{B} at a point $r \in \dot{B}$ (possibly coincident with q but different from p). Thus $p < r$, $(r, q) \in J^+$, hence $p < r$ and $(r, s) \in K^+$. If $q \in B$, since $(q, s) \in R^+ \setminus J^+$, there is $r \in \dot{B}$ such that $q < r$ and $(r, s) \in K^+$, moreover, since $p \leq q$, $p < r$. Since the searched conclusions holds for every B , $(p, s) \in R^+$.

Closure: let $(p_n, q_n) \rightarrow (p, q)$, $(p_n, q_n) \in R^+$. Assume, by contradiction, that $(p, q) \notin R^+$, then $p \neq q$ as $J^+ \subset R^+$. Without loss of generality we can assume two cases: (a) $(p_n, q_n) \in J^+$ for all n ; (b) $(p_n, q_n) \notin J^+$ for all n .

(a) Let $B \ni p$ be an open relatively compact set. For sufficiently large n , $p_n \neq q_n$ and $p_n \in B$. By the limit curve theorem [32] either there is a limit continuous causal curve joining p to q , and thus $p < q$ (a contradiction), or there is a future inextendible continuous causal curve σ^p starting from p such that for every $p' \in \sigma^p$, $(p', q) \in \overline{J^+}$. Since (M, g) is non-total imprisoning, σ^p intersects \dot{B} at some point r . Thus $p < r$ and since $(r, q) \in \overline{J^+} \subset K^+$ we have $(p, q) \in R^+$, a contradiction.

(b) Let $B \ni p$ be an open relatively compact set. For sufficiently large n , $p_n \neq q_n$ and $p_n \in B$. Since $(p_n, q_n) \in R^+$ there is $r_n \in \dot{B}$, $p_n < r_n$, and $(r_n, q_n) \in K^+$. Without loss of generality we can assume $r_n \rightarrow r \in \dot{B}$, so that $(r, q) \in K^+$. Arguing as in (a) either $p < r$ (and $(r, q) \in K^+$) or there is $r' \in \dot{B}$ such that $p < r'$ and $(r', r) \in \overline{J^+} \subset K^+$, from which it follows $(r', q) \in K^+$. Because of the arbitrariness of B , $(p, q) \in R^+$, a contradiction. \square

Corollary 1. *Let (M, g) be non-total imprisoning. The spacetime (M, g) is K -causal if and only if $x < y$ and $(y, x) \in K^+$ implies $x = y$.*

Proof. Since $J^+ \subset K^+$ to the right it is trivial. To the left, assume (M, g) is not K -causal, then there are $z, y \in M$, $z \neq y$ such that $(z, y) \in K^+$ and $(y, z) \in K^+$. If $(z, y) \in J^+$ we have finished with $x = z$. Otherwise, let $B \ni z$ be an open relatively compact set. By lemma 3 there is a point $x \in \dot{B}$ such that $z < x$ and $(x, y) \in K^+$ (and thus $(x, z) \in K^+$) which implies $z = x \in \dot{B}$, a contradiction. \square

Lemma 4.

- (a) *Let \tilde{t} be a continuous function such that $x \leq y \Rightarrow \tilde{t}(x) \leq \tilde{t}(y)$. If $(p, q) \in K^+$ then $\tilde{t}(p) \leq \tilde{t}(q)$.*
- (b) *Let t be a time function on (M, g) . If $(p, q) \in K^+$ then $p = q$ or $t(p) < t(q)$.*

Proof. Proof of (a). Consider the relation

$$\tilde{R}^+ = \{(p, q) \in K^+ : \tilde{t}(p) \leq \tilde{t}(q)\}.$$

Clearly $J^+ \subset \tilde{R}^+ \subset K^+$ and \tilde{R}^+ is transitive.

Let us prove that \tilde{R}^+ is closed. If $(x_n, z_n) \in \tilde{R}^+$ is a sequence such that $(x_n, z_n) \rightarrow (x, z)$, then passing to the limit $\tilde{t}(x_n) \leq \tilde{t}(z_n)$ and using the continuity

of \tilde{t} we get $\tilde{t}(x) \leq \tilde{t}(z)$, moreover since K^+ is closed, $(x, z) \in K^+$, which implies $(x, z) \in \tilde{R}^+$, that is \tilde{R}^+ is closed.

Since $J^+ \subset \tilde{R}^+ \subset K^+$, and \tilde{R}^+ is closed and transitive, by using the minimality of K^+ it follows that $\tilde{R}^+ = K^+$. As a consequence, if $(p, q) \in K^+$ then $\tilde{t}(p) \leq \tilde{t}(q)$.

Proof of (b). By lemma 2, since t is a time function (M, g) is strongly causal and thus non-total imprisoning. Consider the relation

$$R^+ = \{(p, q) \in K^+ : p = q \text{ or } t(p) < t(q)\}.$$

Clearly $J^+ \subset R^+ \subset K^+$ and R^+ is transitive. Let us prove that R^+ is closed by keeping in mind the just obtained result given by (a). Let $(p_n, q_n) \in R^+ \subset K^+$ be a sequence such that $(p_n, q_n) \rightarrow (p, q)$. As K^+ is closed, $(p, q) \in K^+$. If, by contradiction, $(p, q) \notin R^+$ then $(p, q) \notin J^+$, thus by lemma 3, chosen an open relatively compact set $B \ni p$ there is $r \in \bar{B}$, with $p < r$, $(r, q) \in K^+$, thus $t(p) < t(r) \leq t(q)$ and hence $(p, q) \in R^+$, a contradiction.

Since $J^+ \subset R^+ \subset K^+$, and R^+ is closed and transitive, by using the minimality of K^+ it follows that $R^+ = K^+$. As a consequence, if $(p, q) \in K^+$ then either $p = q$ or $t(p) < t(q)$. \square

Theorem 6. *In a K -causal spacetime the continuous K^+ -utilities are the time functions.*

Proof. A K^+ -utility is a function u which satisfies (i) $(x, y) \in K^+ \Rightarrow u(x) \leq u(y)$ and (ii) $(x, y) \in K^+$ and $(y, x) \notin K^+ \Rightarrow u(x) < u(y)$. Since the spacetime is K -causal this condition is equivalent to $(x, y) \in K^+ \Rightarrow x = y$ or $u(x) < u(y)$. Thus by lemma 4 point (b), every time function is a continuous K^+ -utility. Conversely, in a K -causal spacetime a continuous K^+ -utility satisfies $x < y \Rightarrow (x, y) \in K^+ \setminus \Delta \Rightarrow u(x) < u(y)$ and hence it is a time function. \square

Theorem 7. *A spacetime is K -causal if and only if it admits a time function (as a consequence time functions are always K^+ -utilities). In this case, denoting with \mathcal{A} the set of time functions we have that the partial order K^+ can be recovered from the time functions, that is*

$$(x, y) \in K^+ \Leftrightarrow \forall t \in \mathcal{A}, t(x) \leq t(y). \quad (3)$$

Proof. Assume that the spacetime admits a time function then it is K -causal, that is K^+ is antisymmetric, indeed otherwise there would be $p, q \in M$, $p \neq q$, such that $(p, q) \in K^+$ and $(q, p) \in K^+$. By lemma 4(b), $t(p) < t(q) < t(p)$, a contradiction. By theorem 6 we also infer that the time function is a continuous K^+ -utility.

Assume that the spacetime is K -causal then by Levin's theorem it admits a continuous K^+ -utility which by theorem 6 is a time function.

The last statement is also an application of Levin's theorem. \square

Actually Levin's theorem states something more because it applies also to the case in which K -causality does not hold. However, in this case the K^+ -utilities are not time functions. Nevertheless, we have the following

Lemma 5. *In a chronological spacetime in which $\overline{J^+}$ is transitive (that is $K^+ = \overline{J^+}$) the continuous K^+ -utilities are also continuous I^+ -utilities, that is, they are also semi-time functions.*

Proof. Let u be a K^+ -utility, since the spacetime is chronological we have only to prove $(x, y) \in I^+ \Rightarrow u(x) < u(y)$. The hypothesis is (i) $(x, y) \in \overline{J^+} \Rightarrow u(x) \leq u(y)$ and (ii) $(x, y) \in \overline{J^+}$ and $(y, x) \notin \overline{J^+} \Rightarrow u(x) < u(y)$. Note that if $(x, y) \in I^+$ then $(y, x) \notin \overline{J^+}$ because the relation I^+ is open and the spacetime is chronological, thus $(x, y) \in I^+ \Rightarrow (x, y) \in \overline{J^+}$ and $(y, x) \notin \overline{J^+} \Rightarrow u(x) < u(y)$ which is the thesis. \square

As a consequence we are able to clarify that the consequences of Levin's theorem are actually stronger than those of Peleg's theorem, as we can now infer from theorem 3

Theorem 8. *A chronological spacetime in which $\overline{J^+}$ is transitive admits a semi-time function.*

Of course it actually admits a continuous K^+ -utility which is a stronger concept than that of semi-time function. We have expressed the theorem in this form for the sake of comparison with theorem 5.

By using Levin's theorem and the smoothing result for time functions [4] it is possible to give another proof of the equivalence between K -causality and stable causality

Theorem 9. *K -causality coincides with stable causality.*

Proof. The proof that stable causality implies K -causality goes as usual. The thesis follows from $K^+ \subset J_S^+$, because J_S^+ is closed, transitive and contains J^+ while K^+ is by definition the smallest relation with this property. Thus this direction follows from the equivalence between the antisymmetry of J_S^+ and stable causality, the antisymmetry condition being inherited by the inclusion of relations.

For the other direction K -causality implies the existence of a time function, thus the existence of a temporal function and hence stable causality. \square

4. Time orderings

This section is independent of the previous one. Here the representation theorem for K^+ (or J_S^+) through the time functions is proved again without the help of Levin's theorem but using the results of [34]. I gave this proof before discovering the connection with utility theory. It is quite short uses in an essential way the equivalence between K -causality and stable causality. In the last part of the proof I also use the smoothability results of [4] in order to generalize to temporal functions the representation theorem. This improvement is important because it allows us to make a connection with 'observers' on spacetime provided we model them with congruences of timelike curves.

Given a time function on spacetime let us introduce the total preorder

$$T^+[t] = \{(p, q) \in M \times M : t(p) \leq t(q)\}. \quad (4)$$

Any such preorder, here called *time ordering*, extends J^+ according to the definition of section 2, in particular $J^+ \subset T^+[t]$. The relation $T^+[t]$ is closed because t is continuous. If t is temporal then we shall say that the time ordering $T[t]$ is also a *temporal ordering*.

Note that the relation $T^+[t]$ is invariant under monotonous time reparametrizations, that is, if f is increasing $f(t(\cdot))$ is a time function and

$$T^+[f(t)] = T^+[t].$$

In other words, the relation $T^+[t]$ keeps the information on the *simultaneity* convention associated to the time function t , but it is insensitive to the actual values of the time intervals $t(q) - t(p)$, $p, q \in M$.

As a matter of convention, in the next intersections if the index sets \mathcal{A} or \mathcal{B} are empty then the intersection is the whole ambient space $M \times M$.

Theorem 10. *In every spacetime*

$$K^+ \subset J_S^+ \subset \bigcap_{t \in \mathcal{A}} T^+[t] \subset \bigcap_{t \in \mathcal{B}} T^+[t].$$

In a stably causal spacetime

$$K^+ = J_S^+ = \bigcap_{t \in \mathcal{A}} T^+[t] = \bigcap_{t \in \mathcal{B}} T^+[t].$$

Proof. The first inclusion is well known while the latter is obvious because $\mathcal{B} \subset \mathcal{A}$. If, by contradiction, $J_S^+ \subset \bigcap_{t \in \mathcal{A}} T^+[t]$ does not hold then there is $(p, q) \in J_S^+ \setminus \bigcap_{t \in \mathcal{A}} T^+[t]$. In particular \mathcal{A} is not empty and there is a time function t such that $t(p) > t(q)$. Now, note that $J_S^+ \cap \{\bigcap_{t \in \mathcal{A}} T^+[t]\} \subsetneq J_S^+$ being the intersection of closed and transitive relations which contain J^+ , shares all these same properties. As a consequence $K^+ \neq J_S^+$, but it is known [34] that in a stably causal spacetime $K^+ = J_S^+$, thus the spacetime is not stably causal although there is a time function t , a contradiction.

Let (M, g) be a stably causal spacetime. The equality $K^+ = J_S^+$ has been proved in [31, 34]. Let us prove $\bigcap_{t \in \mathcal{A}} T^+[t] \subset J_S^+$. By contradiction, assume it does not hold, then there is a pair $(p, q) \in \bigcap_{t \in \mathcal{A}} T^+[t] \setminus J_S^+$. Recall [31, Lemma 3.3] that $J_S^+ = \bigcap_{g' > g} \overline{J_{g'}^+}$. Since $(p, q) \notin J_S^+$ there is $g' > g$ such that $p \notin \overline{J_{g'}^+}(q)$ and (M, g') is causal.

Let $A \ni p$ be an open set such that $A \cap \overline{J_{g'}^+}(q) = \emptyset$. We are going to construct a time function \hat{t} such that $\hat{t}(p) > \hat{t}(q)$, a contradiction with $(p, q) \in \bigcap_{t \in \mathcal{A}} T^+[t]$. Basically we are going to use Hawking's averaging technique [19, Prop. 6.4.9]. We introduce a volume measure μ as in [19, Prop. 6.4.9] so that $\mu(M)$ is finite. We can find a family of Lorentz metrics $h(a)$, $a \in [0, 3]$, such that points (1)-(3) in that proof are satisfied and $h(3) = g'$. Then we construct a continuous function $\bar{\theta}(x) = \int_1^2 \theta(x, a) da$ where $\theta(x, a) = \mu(I_{h(a)}^-(x))$ as done there. However, here we make just a little change. The measure μ is taken with support in $A \cap I_g^-(p)$. As a consequence the function $\bar{\theta}$ is continuous and non-decreasing over every future directed causal curve while in Hawking's construction it is increasing. Let t be a time function. The continuous function $\hat{t} = t + \bar{\theta}$ is a time function and $\hat{t}(q) = t(q)$ while $\hat{t}(p) = t(p) + \mu(M)$. By choosing $\mu(M) > t(q) - t(p)$ we get the thesis.

It remains to prove the inclusion $\bigcap_{t \in \mathcal{A}} T^+[t] \supset \bigcap_{t \in \mathcal{B}} T^+[t]$. By contradiction, suppose it does not hold then there is a pair $(p, q) \in \bigcap_{t \in \mathcal{B}} T^+[t] \setminus \bigcap_{t \in \mathcal{A}} T^+[t]$.

In other words for every temporal function t (there is at least one temporal function τ because (M, g) is stably causal [4]), we have $t(p) \leq t(q)$, but there is a time function \hat{t} such that $\hat{t}(p) > \hat{t}(q)$. Consider the (acausal) partial Cauchy hypersurface $S = \hat{t}^{-1}(\hat{t}(p))$, see figure 1. The set S does not intersect q , let $N = M \setminus \{q\}$ so that S is a partial Cauchy hypersurface for $(N, g|_N)$. Let $D_N(S)$ be the Cauchy development of S on $(N, g|_N)$ and $H_N^+(S)$ and $H_N^-(S)$ the future and past Cauchy horizons. We have $S \cap H_N^+(S) = \emptyset$ because if $r \in S \cap H_N^+(S)$ then, as $H_N^+(S)$ is generated by past inextendible lightlike geodesics on N , there would be a past inextendible geodesic with future endpoint r . No other point of the geodesic can belong to S because of its acausality, but since $H_N^+(S) \setminus S \subset I_N^+(S)$ we get a contradiction with the achronality of S . Analogously, $S \cap H_N^-(S) = \emptyset$.

As a consequence the set $\text{Int}D_N(S)$ is non-empty and being globally hyperbolic it is diffeomorphic to $\mathbb{R} \times S$ where the slices diffeomorphic to S are the level sets of a temporal function t' on the spacetime $\text{Int}D_N(S)$ with the induced metric (see [4]). Choosing a, b , $a < b$, so that $b < t'(p)$ and $a < b + \tau(p) - \tau(q)$, we construct a function t'' on $\text{Int}D_N(S)$ so that $t'' = t'$ at those points where $a \leq t' \leq b$, $t'' = b$ at those points such that $t' \geq b$, and $t'' = a$ at those points where $t' \leq a$. Note that $t''(p) = b$. Clearly t'' has past directed timelike gradient for $a < t'' < b$ but there is a discontinuity in the gradient for $t'' = a$ or $t'' = b$. However, a smooth monotonous reparametrization $t''' = f(t'')$ exists which sends a to a , b to b , and makes the gradient everywhere continuous, timelike on $a < t''' < b$ and vanishing for $t''' \leq a$, and $t''' \geq b$. A possible choice is

$$t''' = -\frac{b-a}{2} \cos\left[\pi \frac{t''-a}{b-a}\right] + \frac{b+a}{2}, \quad \text{for } a \leq t'' \leq b, \quad t''' = t'' \text{ elsewhere.}$$

The function t''' can be extended in a smooth way to M by setting $t''' = b$ on $\hat{t}^{-1}((\hat{t}(p), +\infty)) \setminus \text{Int}D_N(S)$ and $t''' = a$ on $\hat{t}^{-1}((-\infty, \hat{t}(p))) \setminus \text{Int}D_N(S)$. In particular, since $q \notin \text{Int}D_N(S)$ and $\hat{t}(p) > \hat{t}(q)$ we have $t'''(q) = a$. The function $\check{t} = \tau + t'''$ is a temporal function and $\check{t}(q) = \tau(q) + a < \tau(p) + b = \check{t}(p)$, a contradiction. \square

It must be remarked that to every temporal function t there corresponds a flow generated by the future directed timelike unit vector $u = -\nabla t / \sqrt{-g(\nabla t, \nabla t)}$. The generated congruence of timelike curves represents an extended reference frame so that every curve of the congruence is identified with an observer “at rest in the frame”. The flow is orthogonal to the slices $t = \text{const.}$ which therefore are the natural simultaneity slices as they would be obtained by the observers at rest in the frame by a local application of the Einstein’s simultaneity convention [29, 40, 26]. This observation shows that the temporal functions, at least in principle, can be physically realized through a well defined operational procedure. The above theorem then states that while observers living in different extended reference frames may disagree on which event of a pair comes “before” or “after” the other, according to their own time function, they certainly agree whenever the pair of compared events belong to the K^+ (Seifert) relation, and in fact only for those type of pairs. In other words the K^+ (Seifert) relation provides that set of pairs of events for which all the observers agree on their temporal order.

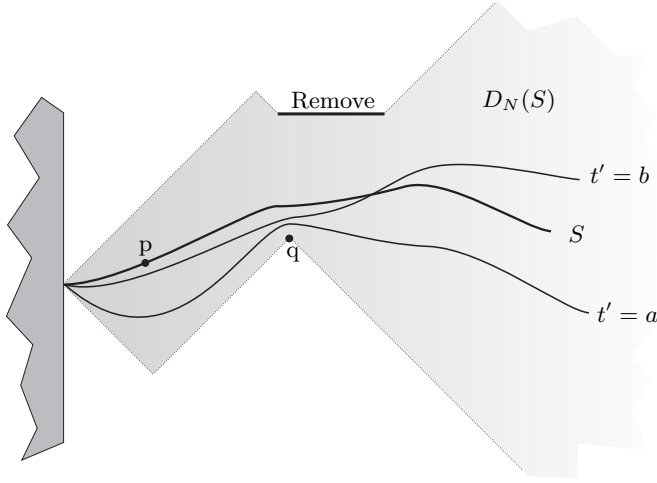


Fig. 1. The last argument in the proof of theorem 10.

Eq. (3) can be rewritten in the equivalent form

$$K^+ = \bigcap_{t \in \mathcal{A}} T^+[t], \quad (5)$$

thus we have just obtained an alternative proof for the same equation.

This result allows us to establish those circumstances in which the chronological or causal relation can be recovered from the knowledge of the time or temporal functions.

Recall that a spacetime is *causally easy* if it is strongly causal and \overline{J}^+ is transitive [34]. Recall also that a causally continuous spacetime is a spacetime which is distinguishing and reflective. Finally a spacetime is causally simple [5] if it is causal and $\overline{J}^+ = J^+$. We have causal simplicity \Rightarrow causal continuity \Rightarrow causal easiness \Rightarrow K -causality.

By definition of causal easiness $K^+ = \overline{J}^+$, thus as $\overline{I}^+ = \overline{J}^+$, we easily find

Proposition 1. *In a causally easy spacetime $I^+ = \text{Int} \bigcap_t T^+[t]$, and in a causally simple spacetime $J^+ = \bigcap_t T^+[t]$ where the intersections are with respect the sets of time or the temporal functions.*

5. Conclusions

The concept of causal influence is more primitive, and in fact more intuitive, than that of time. General relativistic spacetimes have by definition a causal structure but may lack a time function, namely a continuous function which respects the notion of causal precedence (i.e. if a influences b then the time of a is less than that of b).

In this work we have recognized the mathematical coincidence between the problem of the existence of a (semi-)time function on spacetime in the relativistic physics field and the problem of the existence of a utility function for an agent in

microeconomics. From these problems two so far independent lines of research arose which, as we noted, often passed through the very same concepts. Remarkably, some results obtained in one field were not rediscovered in the other, a fact which has allowed us to use Peleg's and Levin's theorems to reach new results concerning the existence of (semi-)time functions in relativity.

In particular, we have proved that a chronological spacetime in which $\overline{J^+}$ is transitive (for instance a reflective spacetime) admits a semi-time function. Also in a K -causal spacetime the existence of a time function follows solely from the closure and antisymmetry of the K^+ relation. In the other direction we have proved without the help of smoothing techniques, that the existence of a time function implies K -causality. We have also given a new proof of the equivalence between K -causality and stable causality by using Levin's theorem and smoothing techniques.

Finally, we have shown in two different ways that in a K -causal (i.e. stably casual) spacetime the K^+ (i.e. Seifert) relation can be recovered from the set of time or temporal functions allowed by the spacetime. This result singles out the K^+ relation as one of the most important for the development of causality theory.

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